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The entropy production rate is calculated for an interchange driven turbulence both in fluid and kinetic regimes. This calculation provides a rigorous way to define thermodynamical forces and fluxes. It is found that the forces are the gradients of density and temperature normalized to their "canonical" values, which are Lagrangian invariants of the flow. This formulation is equivalent to expressing the fluxes in terms of "curvature pinches," where the curvature pinches are proportional to the logarithmic gradient of canonical profiles. Off diagonal terms in the transport matrix are found, which correspond to thermodiffusion and its Onsager symmetrical contribution to the heat flux. Hence, if thermodiffusion is significant, a heat pinch due to the density gradient also exists. The entropy production rate is found to be minimum when the profiles are equal to their canonical values. This property yields a generalized form of profile stiffness. However, a state where all profiles match their canonical values is not attainable because it is linearly stable. [DOI: 10.1063/1.1951667]

I. INTRODUCTION

The structure of the turbulent transport matrix in magnetized plasmas is of utmost importance for fusion. Particle transport in tokamaks is certainly a good example. In absence of internal fueling, the density profile is fully determined by the nondiagonal term in the particle flux, called particle pinch (see, for instance, the overview by Wagner and Stroth[1]). Since density peaking leads to enhanced fusion power, this subject is clearly of interest for the operation of a next step device. Also the study of particle pinch brings some insight into the mechanisms which underlie turbulent transport. Hence the identification and characterization of the particle pinch is traditionally an active field of research.[2–6] The question of heat pinches, i.e., nondiagonal terms in the heat flux, has been also debated. Heat pinch is more difficult to demonstrate than particle pinch because the heat source is never fully peripheral in a tokamak. Evidence of heat pinch has been found in experiments using steady[7] and modulated[8] off-axis electron heating. Two types of nondiagonal terms in the transport matrix have been identified in the literature. On the one hand, the conservation of Lagrangian invariants along the turbulent flow leads to a ratio of "curvature" pinch velocity to diffusion coefficient that is determined by the magnetic configuration only. Curvature pinches were in fact introduced in the frame of the "turbulence equi-partition" (TEP) theory.[9–11] Numerical simulations of two-dimensional (2D) interchange turbulence showed that Lagrangian invariants are conserved, leading to pinches.[12] On the other hand, other contributions come from off-diagonal thermodynamical forces. If the analysis is restricted to particle and heat fluxes, a thermodiffusion term is therefore expected in the particle flux, which is proportional to the logarithmic gradient of the temperature.[13–16] Similarly a contribution proportional to the density gradient is expected in the heat flux.[17] Several numerical simulations showed that turbulent pinches take place in tokamak plasmas,[18] including recent gyrokinetic calculations.[19,20] Also a conventional quasilinear theory predicts that both curvature and thermodiffusion effects appear in particle transport. This feature has been found in the Weiland[18] and GLF23[21] models. A reversal of thermodiffusion is expected when the average phase velocity of fluctuations changes sign in the frame of reference where the radial electric field is zero.[21,22] Since then, many experimental results have been obtained, which suggest that curvature pinch is the main contribution in many cases.[23–28] However, some evidence of thermodiffusion also exists, and it also found that its sign changes when moving from a turbulence dominated by ion modes to a trapped electron mode turbulence.[29] Particle "pump-out" with electron cyclotron resonant heating has been explained by a reversal of thermodiffusion at low collisionality.[30] At larger collisionality, theory predicts a decrease of turbulent pinches, as observed experimentally.[21] A density flattening is sometimes observed with radio frequency heating at high density, but it is attributed to an increase of the diffusion coefficient while the Ware pinch velocity remains constant (or gets smaller).[30]

Several remarks have been formulated concerning this description of transport. First, curvature is not a thermodynamical force, in contrast with the density and temperature gradients. Moreover, it is not always easy to separate these two effects in a unique way. This leads to some confusion in the terminology depending on the definition that has been chosen. Second, consistency between particle and heat fluxes is crucial since Onsager symmetries are expected to appear when using a quasilinear theory.[31] Curvature pinches are not constrained by Onsager symmetries since these are geometry effects. This is why it is quite important to separate properly the two effects. Finally, the transport matrix must be consistent with the second principle of thermodynamics. This condition is not trivially respected when some of the contributions to the fluxes are pointed in the direction of the gradient.

A way to answer all these questions and to ensure the consistency with thermodynamics is to calculate the entropy production rate. Positivity of the entropy production rate guarantees that the second principle is respected. It also allows defining rigorously the thermodynamical forces, including the geometrical effects. Finally, it gives some insight in the order of magnitude of the various terms. The aim of this paper is to calculate the entropy production rate in the case of an electrostatic interchange turbulence in a tokamak. This is done using both fluid and kinetic descriptions. For passing particles, the fluid formulation corresponds to the early version of the Isichenko–Yankov[32] calculation, and also to the Weiland model[18] in its initial form. It is also extended to trapped particles.

The paper is organized as follows. Section II presents the basic fluid equations, the main instabilities and the fluxes. The entropy production rate is calculated in Sec. III for fluid theory and in Sec. IV for the kinetic case. Implications for the pinches and considerations on Onsager symmetry are given in Sec. V. A conclusion is given in Sec. VI.

II. FLUXES AND THERMODYNAMICAL FORCES

A. Fluid equations

Plasma electrostatic turbulence is computed by coupling Maxwell's equations (restricted here to a quasi-electroneutrality constraint) and the plasma response. Here we use the Braginskii equations[33] in the collisionless limit. The result of a kinetic approach will be seen later on. Each species s is characterized by a continuity equation

$$d_t n_s = - n_s \nabla \cdot \mathbf{V_s}, \tag{1a}$$

a force balance equation

$$n_s m_s d_t \mathbf{V_s} = - \nabla p_s - \nabla \cdot \boldsymbol{\pi}_\mathbf{s} + n_s e_s (\mathbf{E} + \mathbf{V_s} \times \mathbf{B}), \tag{1b}$$

and a heat equation

$$d_t p_s = - \varsigma p_s \nabla \cdot \mathbf{V_s} - (\varsigma - 1) \nabla \cdot \mathbf{q_s} - (\varsigma - 1) \boldsymbol{\pi}_\mathbf{s} : \nabla \mathbf{V_s}, \tag{1c}$$

where n_s, $\mathbf{V_s}$, and p_s are the density, velocity, and pressure, $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields, $d_t = \partial_t + \mathbf{V_s} \cdot \boldsymbol{\nabla}$ is the Lagrangian derivative, e_s and m_s are the charge and mass, and $\varsigma = 5/3$ is the adiabatic index. The adiabatic index will be considered in the following as a free parameter, such that $\varsigma > 1$. The heat flux $\mathbf{q_s}$ is restricted to its diamagnetic component:

$$\mathbf{q}_s^* = \varsigma \frac{p_s}{e_s} \left(\frac{\mathbf{B_{eq}}}{B_{eq}^2} \times \nabla T_s \right). \tag{2}$$

The stress tensor $\boldsymbol{\pi}_\mathbf{s}$ is detailed in the Braginskii paper and includes finite Larmor radius (FLR) effects. In the electrostatic limit ($\beta \to 0$) passing electrons are assumed to be massless. Using the projection of Eq. (1b) along field lines, this means that their density response is of the Boltzmann type

$$n_e - n_{e,eq} = f_c n_{e,eq} \frac{e(\phi - \phi_{eq})}{T_{e,eq}}. \tag{3}$$

Here e is the proton charge and f_c is the fraction of passing electrons ($f_c = 1 - f_t$, where f_t is the fraction of trapped electrons). Equation (1b) indicates that the perpendicular velocity is the sum of an $E \times B$ drift and diamagnetic drift velocities at lowest order in normalized Larmor radius, i.e.,

$$\mathbf{V}_{\perp s} = \frac{\mathbf{B}_{eq}}{B_{eq}^2} \times \nabla \phi + \frac{\mathbf{B}_{eq}}{B_{eq}^2} \times \frac{\nabla p_s}{n_s e_s}. \tag{4}$$

The stress tensor introduces further corrections in the perpendicular velocity, corresponding to FLR effects (not detailed here). Since the perpendicular velocity is prescribed by Eq. (4), each species is described by three scalar fields: density, parallel velocity, and pressure. In fact, trapped electrons exhibit a zero average velocity due to their bounce motion. Fluid equations must be rederived in this case from the bounce average kinetic equation. In practice, the same result is found, except that there is no parallel velocity, and the divergence of the $E \times B$ drift must be replaced by the precession frequency of trapped electrons (see Appendix A). Also, calculations are done in the frame of reference where the radial electric field, assumed to be shearless, vanishes. In the 2D limit, one gets the following set of equations on density and pressure:

$$d_t n_s = - n_s \nabla \cdot \boldsymbol{\nu}_\mathbf{E} - \nabla \cdot \boldsymbol{\Gamma}_\mathbf{s}^* = - \boldsymbol{\kappa}_\mathbf{s} \cdot \left(n_s \nabla \phi + \frac{\nabla p_s}{e_s} \right), \tag{5}$$

$$\begin{aligned} d_t p_s &= - \varsigma p_s \nabla \cdot \boldsymbol{\nu}_\mathbf{E} - \nabla \cdot \mathbf{Q}_\mathbf{s}^* \\ &= - \varsigma \boldsymbol{\kappa}_\mathbf{s} \cdot \left(p_s \nabla \phi + \frac{1}{e_s} \nabla \frac{p_s^2}{n_s} \right), \end{aligned} \tag{6}$$

where

$$\boldsymbol{\Gamma}_\mathbf{s}^* = \frac{\mathbf{B_{eq}}}{B_{eq}^2} \times \frac{\nabla p_s}{e_s}, \quad \mathbf{Q}_\mathbf{s}^* = \varsigma T_s \boldsymbol{\Gamma}_\mathbf{s}^* + \mathbf{q}_\mathbf{s}^* \tag{7}$$

and

$$\boldsymbol{\kappa}_\mathbf{s} = \lambda_s \frac{2}{B_{eq}} \frac{\mathbf{B}_{eq}}{B_{eq}} \times \frac{\nabla B_{eq}}{B_{eq}}. \tag{8}$$

Although calculations are easier to derive when using the density and pressure, we anticipate that the right thermodynamical force is the temperature rather than the pressure logarithmic gradient. From now on, we will therefore replace the evolution equation for the pressure by a heat equation, which can be deduced from Eqs. (5) and (6)

$$\begin{aligned} d_t T_s = - \boldsymbol{\kappa}_\mathbf{s} \cdot \Bigg(& (\varsigma - 1) T_s \nabla \phi + (\varsigma - 1) \frac{T_s^2}{e_s} \frac{\nabla n_s}{n_s} \\ & + (2\varsigma - 1) \frac{T_s \nabla T_s}{e_s} \Bigg). \end{aligned} \tag{9}$$

The seminal work of Isichenko and Yankov[32] was done for a 2D interchange turbulence which corresponds to Eqs. (5) and (6) with $\lambda_s = 1/2$. The original Weiland model[18] corresponds to $\lambda_s = 1$. It can be extended to passing particles in a tokamak by stating $\lambda_s = \langle \cos(\eta) + s \eta \sin(\eta) \rangle$, where η is a coordinate along field lines, $s = d \log(q)/d \log(r)$ is the magnetic shear (q is the safety factor) and the bracket indicates an average over the mode structure along the magnetic field. Including trapped particles in the model is done by replacing the curvature drift by the precession frequency of particles (see Appendix A for details). This is equivalent to setting $\lambda_s = 1/4 + 2s/3$. The dependence on magnetic shear is still controversial and is model dependent.[20] This is why the parameter λ_s is considered here as a free parameter with arbitrary dependences.

The following calculations are applicable for any equilibrium magnetic field of the form $\mathbf{B_{eq}} = f(\psi)\nabla\varphi + \nabla\psi \times \nabla\varphi$, where ψ is the poloidal flux and φ the toroidal angle. The principal condition of applicability is that B_{eq} (modulus of the magnetic field) depends on ψ only. This is typically the case of a cylindrical geometry (with periodic conditions along the axis). In a tokamak, this would correspond to the case of flute modes with an effective curvature, or to the case of a strongly ballooned turbulence, or to particles trapped on the low field side. We will illustrate the calculations by using a quasicylindrical equilibrium. The effective curvature is directed in the poloidal direction, $\boldsymbol{\kappa}_\mathbf{s} = -2\lambda_s / B_0 R_0 \mathbf{e}_{\boldsymbol{\theta}}$, where R_0 is the major radius. Here the angle θ refers to the poloidal direction, $d\varphi/d\theta = q$ along field lines) and the radial coordinate will be noted $r(d\psi = -B_0 r dr/q)$. All calculations are done at order one in r/R_0.

B. Canonical profiles and stability

It is interesting to analyze the equations for density and temperature (5) and (9) in the case where diamagnetic effects are ignored, i.e., when only the terms proportional to the perturbed electric field are kept in the right-hand side (RHS) of these equations. For particles, this corresponds to the case where the perturbed temperature is small (i.e., without thermodiffusion). Equations (5) and (9) can be rewritten in the compact form[32]

$$d_t \bar{n}_s = 0, \tag{10}$$

$$d_t \bar{T}_s = 0, \tag{11}$$

where

$$\bar{n}_s = n_s I_s, \quad \bar{T}_s = T_s I_s^{\varsigma - 1} \tag{12}$$

and

$$\frac{\nabla I_s}{I_s} = -2\lambda_s \frac{\nabla B_{eq}}{B_{eq}}. \tag{13}$$

This relation allows to construct I_s for arbitrary geometry (I_s is defined up to a multiplicative constant that is chosen such that $I_s = 1$ on the magnetic axis). This is nothing but the result predicted by the TEP theory. For given statistical properties of the $E \times B$ drift velocity field, Eqs. (10) and (11) imply that the transported quantities are the normalized quantities $\bar{n}_s$ and $\bar{T}_s$, and not the actual density and temperature. At this stage, this is still a very general result that applies independently of the nature of transport, i.e., diffusive or anomalous. In the particular case where the statistics is Gaussian and conditions of applicability of the quasilinear theory are met, the transport is diffusive. In absence of sources and sinks, the solutions of transport equations are therefore $\bar{n}_s = n_{s0}$ and $\bar{T}_s = T_{s0}$. The corresponding profiles are

$$n_{s,can} = n_{s0}[I_s]^{-1}, \quad T_{s,can} = T_{s0}[I_s]^{-(\varsigma - 1)}. \tag{14}$$

They will be called "canonical" profiles, following the definition of Isichenko and Yankov[32] (in the latter work, which corresponds to pure 2D interchange turbulence, $\lambda_s = 1/2$). When recast into transport equations on n_s and T_s, this formulation is equivalent to introducing "curvature" pinch velocities such that $V_s/D_s = \partial_r I_s / I_s$ for density and $V_{Ts}/\chi_s = (\varsigma - 1)\partial_r I_s / I_s$ for temperature, where χ_s is the heat diffusivity. However, this presentation is somewhat misleading since in fact the curvature term $\partial_r I_s / I_s$ (equals to $2\lambda_s / R_0$ in quasicylindrical geometry) is not a thermodynamical force. A better presentation is to use the gradients of $\bar{n}_s$ and $\bar{T}_s$, i.e., of $n_s / n_{s,can}$ and $T_s / T_{s,can}$, as the actual forces. Equations (5) and (9) can restated as follows, using these new functions:

$$\frac{d_t \bar{n}_s}{\bar{n}_s} = \frac{T_s}{e_s}\left(\frac{\mathbf{B}_{eq}}{B_{eq}^2} \times \frac{\nabla I_s}{I_s}\right) \cdot \left[\frac{\nabla \bar{n}_s}{\bar{n}_s} + \frac{\nabla \bar{T}_s}{\bar{T}_s}\right], \tag{15}$$

$$\frac{d_t \bar{T}_s}{\bar{T}_s} = \frac{T_s}{e_s}\left(\frac{\mathbf{B}_{eq}}{B_{eq}^2} \times \frac{\nabla I_s}{I_s}\right) \cdot \left[(\varsigma - 1)\frac{\nabla \bar{n}_s}{\bar{n}_s} + (2\varsigma - 1)\frac{\nabla \bar{T}_s}{\bar{T}_s}\right], \tag{16}$$

where the relation

$$\boldsymbol{\kappa}_s = -\frac{\mathbf{B}_{eq}}{B_{eq}^2} \times \frac{\nabla I_s}{I_s} \tag{17}$$

has been used. Equations (15) and (16) can be reformulated in terms of density and pressure evolution equations

$$d_t \bar{n}_s = \frac{1}{e_s}\left(\frac{\mathbf{B}_{eq}}{B_{eq}^2} \times \frac{\nabla I_s}{I_s^{\varsigma}}\right) \cdot \nabla \bar{p}_s, \tag{18}$$

$$d_t \bar{p}_s = \frac{\varsigma}{e_s}\left(\frac{\mathbf{B}_{eq}}{B_{eq}^2} \times \frac{\nabla I_s}{I_s^{\varsigma}}\right) \cdot \nabla \left[\frac{\bar{p}_s^2}{\bar{n}_s}\right]. \tag{19}$$

For now on, we will use a quasicylindrical geometry. The following expression of I_s is then obtained:

$$I_s = e^{(2/R_0)\int_0^r \lambda_s dr}. \tag{20}$$

A linear development of Eqs. (5) and (9) leads to the following expressions for density and temperature perturbations:

$$\begin{pmatrix} \dfrac{\bar{n}_{s,\mathbf{k}\omega}}{\bar{n}_{s,eq}} \\ \dfrac{1}{\varsigma - 1}\dfrac{\bar{T}_{s,\mathbf{k}\omega}}{\bar{T}_{s,eq}} \end{pmatrix} = -\frac{1}{D_{s,\mathbf{k}\omega}} \frac{e_s \tilde{\phi}_{\mathbf{k}\omega}}{T_{s,eq}} \begin{pmatrix} \omega - (2\varsigma - 1)\omega_{ds} & \omega_{ds} \\ \omega_{ds} & \dfrac{\omega - \omega_{ds}}{\varsigma - 1} \end{pmatrix} \begin{pmatrix} \omega^*_{\bar{n}_s} \\ \omega^*_{\bar{T}_s} \end{pmatrix}, \tag{21}$$

where the wave vector $\mathbf{k}$ is defined as $\mathbf{k} = k_\theta \mathbf{e}_{\boldsymbol{\theta}} + k_\varphi \mathbf{e}_{\boldsymbol{\varphi}}$ ($k_\theta = m/r$, $k_\varphi = n/R_0$, and $n = -mq$ since we restrict the calculation to flute modes) and

$$D_{s,\mathbf{k}\omega} = \omega^2 - 2\varsigma\omega_{ds}\omega + \varsigma\omega_{ds}^2 \tag{22}$$

and

$$\omega_{ds} = -k_\theta \frac{T_{s,eq}}{e_s B_{eq}} \frac{dI_s}{I_s dr}, \quad \omega_{\bar{n}_s}^* = k_\theta \frac{T_{s,eq}}{e_s B_{eq}} \frac{d\bar{n}_{s,eq}}{\bar{n}_{s,eq} dr},$$
$$\omega_{\bar{T}_s}^* = k_\theta \frac{T_{s,eq}}{e_s B_{eq}} \frac{d\bar{T}_{s,eq}}{\bar{T}_{s,eq} dr}, \tag{23}$$

where the label eq indicate the $m=0$, $n=0$ components of the density and temperature. The link between the extended diamagnetic frequencies and the conventional ones is

$$\omega_{\bar{n}_s}^* = \omega_{n_s}^* - \omega_{ds}, \quad \omega_{\bar{T}_s}^* = \omega_{T_s}^* - (\varsigma - 1)\omega_{ds}. \tag{24}$$

The formulation equation (21) is still correct for arbitrary geometry with appropriate definitions of curvature and diamagnetic frequencies. Consistency is ensured by the quasineutrality constraint

$$\sum_s n_s e_s = 0. \tag{25}$$

For a mixture of electrons and hydrogenic ions, it reads[18]

$$\frac{f_c}{T_{e,eq}} + \frac{f_t}{T_{e,eq} D_e(\omega)} [\omega_{de}\omega_{\bar{T}_e}^* + (\omega - (2\varsigma - 1)\omega_{de})\omega_{\bar{n}_e}^*] + \frac{1}{T_{i,eq} D_i(\omega)} [\omega_{di}\omega_{\bar{T}_i}^* + (\omega - (2\varsigma - 1)\omega_{di})\omega_{\bar{n}_i}^*] = 0. \tag{26}$$

This equation can be recast into the form

$$1 + \frac{A_{\bar{T}_e} + [\Omega - (2\varsigma - 1)]A_{\bar{n}_e}}{\Omega^2 - 2\varsigma\Omega + \zeta} + \frac{A_{\bar{T}_i} + [\Omega + (2\varsigma - 1)]A_{\bar{n}_i}}{\Omega^2 + 2\varsigma\tau_i\Omega + \varsigma\tau_i^2} = 0, \tag{27}$$

where

$$A_{\bar{T}_e} = \frac{f_t}{f_c}\left(\frac{R_0}{2L_{pe}} - (\varsigma - 1)\lambda_e\right),$$
$$A_{\bar{T}_i} = \frac{\tau_i}{f_c}\left(\frac{R_0}{2L_{pi}} - (\varsigma - 1)\lambda_i\right), \tag{28}$$
$$A_{\bar{n}_e} = \frac{1}{f_c}\left(\frac{R_0}{2L_n} - \lambda_e\right), \quad A_{\bar{n}_i} = \frac{\tau_i}{f_c}\left(\frac{R_0}{2L_n} - \lambda_i\right),$$

and $\tau_i = T_{i,eq}/T_{e,eq}$ and $\Omega = \omega/\omega_{de}$. Note that the canonical density and temperature profiles are such that $A_{\bar{n}_e}, A_{\bar{n}_i}, A_{\bar{T}_e}, A_{\bar{T}_i} = 0$. These conditions are *not* the conditions for marginal stability, except if passing electrons are ignored. The solutions of Eq. (26) have been extensively studied.[18] Depending on the range of gradients, zero, one, or two solutions are found, corresponding to trapped electron and ion temperature gradient modes. For a canonical density profile $A_{\bar{n}_e}, A_{\bar{n}_i} = 0$, the domain of marginal stability can be found analytically as a parametric curve with respect to the frequency. A rather accurate approximation of the threshold of the trapped electron mode (TEM) branch is

$$A_{\bar{T}_e} = \varsigma^2 - \varsigma + \frac{\varsigma - 1}{3\varsigma + 1} A_{\bar{T}_i}. \tag{29}$$

A symmetrical relation is found for the ion temperature gradient (ITG) branch

$$A_{\bar{T}_i} = \varsigma^2 - \varsigma + \frac{\varsigma - 1}{3\varsigma + 1} A_{\bar{T}_e}. \tag{30}$$

Therefore the condition $A_{\bar{n}_e}, A_{\bar{n}_i}, A_{\bar{T}_e}, A_{\bar{T}_i} = 0$ corresponds to stable profiles since the gradients are below the critical thresholds (29) and (30).

C. Fluxes

The transport equations are obtained by averaging Eqs. (18) and (19) over the poloidal and toroidal angles, and fast time scales. It is important to note that the RHS of Eqs. (18) and (19) vanish when averaging over the angles. This procedure leads to the following compact equations:

$$\partial_t \bar{n}_{s,eq} + \frac{1}{r}\partial_r(r\Gamma_s) = 0, \tag{31}$$

$$\frac{1}{\varsigma - 1}\partial_t \bar{p}_{s,eq} + \frac{1}{r}\partial_r(rQ_{Es}) = 0, \tag{32}$$

where the particle and heat fluxes are given by the relations

$$\Gamma_s = \langle \bar{n}_s v_{Er} \rangle = \sum_{\mathbf{k},\omega} \bar{n}_{s,\mathbf{k}\omega} \frac{ik_\theta}{B_{eq}} \phi_{\mathbf{k}\omega}^*, \tag{33}$$

$$Q_{Es} = \frac{1}{\varsigma - 1}\langle \bar{p}_s v_{Er} \rangle = \frac{1}{\varsigma - 1}\sum_{\mathbf{k},\omega} \bar{p}_{s,\mathbf{k}\omega} \frac{ik_\theta}{B_{eq}} \phi_{\mathbf{k}\omega}^*. \tag{34}$$

Here v_{Er} is the radial component of the perturbed $E \times B$ drift velocity. A heat transport equation is obtained by subtracting Eq. (31) from Eq. (32),

$$\frac{1}{\varsigma - 1}\bar{n}_{s,eq}\partial_t \bar{T}_{s,eq} + \frac{1}{r}\partial_r(rQ_s) + \frac{1}{\varsigma - 1}\Gamma_s \partial_r \bar{T}_{s,eq} = 0 \tag{35}$$

where

$$Q_s = Q_{Es} - \frac{1}{\varsigma - 1}\bar{T}_{s,eq}\Gamma_s = \frac{1}{\varsigma - 1}\bar{n}_{s,eq}\langle \bar{T}_s v_{Er} \rangle = \frac{1}{\varsigma - 1}\bar{n}_{s,eq}\sum_{\mathbf{k},\omega} \bar{T}_{s,\mathbf{k}\omega} \frac{ik_\theta}{B_0} \phi_{\mathbf{k}\omega}^*. \tag{36}$$

The flux Q_s is the conductive heat flux, i.e., the difference between the total heat flux Q_{Es} and the convective heat flux $\bar{T}_{s,eq}\Gamma_s/(\varsigma - 1)$. The convective flux is finite for nonzero particle flux. It differs from the "convective" terms (pinch velocities) in the expressions of the particle and conductive heat fluxes.

Quasilinear fluxes are obtained by using the linear expressions for density and temperature fluctuations Eqs. (21) and (22), and be written in the matrix form

$$\begin{pmatrix} \dfrac{\Gamma_s}{\bar{n}_{s,eq}} \\ \dfrac{Q_s}{\bar{n}_{s,eq}\bar{T}_{s,eq}} \end{pmatrix} = -\sum_{\mathbf{k}\omega} i \frac{|\tilde{v}_{E,\mathbf{k}\omega}|^2}{D_{s,\mathbf{k}\omega}} \begin{bmatrix} \omega-(2\varsigma-1)\omega_{ds} & \omega_{ds} \\ \omega_{ds} & \dfrac{\omega-\omega_{ds}}{\varsigma-1} \end{bmatrix} \cdot \begin{pmatrix} \dfrac{\partial_r \bar{n}_{s,eq}}{\bar{n}_{s,eq}} \\ \dfrac{\partial_r \bar{T}_{s,eq}}{\bar{T}_{s,eq}} \end{pmatrix}, \tag{37}$$

where

$$\tilde{v}_{E,\mathbf{k}\omega} = \frac{-ik_\theta \tilde{\phi}_{\mathbf{k}\omega}}{B_0}. \tag{38}$$

In the case where turbulence frequencies are larger than curvature frequencies, one finds that the matrix is diagonal, i.e.,

$$\begin{pmatrix} \Gamma_s \\ Q_s \end{pmatrix} = -\begin{bmatrix} D_{turb} & 0 \\ 0 & \dfrac{1}{\varsigma-1} D_{turb} \end{bmatrix} \cdot \begin{pmatrix} \partial_r \bar{n}_{s,eq} \\ \bar{n}_{s,eq}\partial_r \bar{T}_{s,eq} \end{pmatrix}, \tag{39}$$

where

$$D_{turb} = \pi \sum_{\mathbf{k}\omega} |\tilde{v}_{E,\mathbf{k}\omega}|^2 \delta(\omega). \tag{40}$$

A simple relation is found between the heat conductivity and the diffusion coefficient in this limit, $\chi_s = D_s/(\zeta-1)$, i.e., $\chi_s = 3D_s/2$ for $\zeta=5/3$. In the general case described by Eq. (37), no simple relation exists between these two quantities.

An interesting case is obtained when the time response at scale k^{-1} is broadened by some nonlinear decorrelation term with a characteristic frequency $\Delta\omega_{\mathbf{k}}$. In this case the frequency spectrum of fluctuations is of the form

$$|\tilde{v}_{E,\mathbf{k}\omega}|^2 = |\tilde{v}_{E\mathbf{k}}|^2 \frac{1}{\pi} \frac{\Delta\omega_{\mathbf{k}}}{(\omega-\omega_{\mathbf{k}})^2 + \Delta\omega_{\mathbf{k}}^2}, \tag{41}$$

where $\omega_{\mathbf{k}}$ is a real frequency [for instance, $\omega_{\mathbf{k}} = \varsigma\omega_{ds}$ in the Weiland model, while $\Delta\omega_{\mathbf{k}} = \gamma_{\mathbf{k}}$ (growth rate) and $|\tilde{v}_{E,\mathbf{k}\omega}| \propto \gamma_{\mathbf{k}}$ (Ref. 18)]. The turbulent diffusion coefficient reads

$$D_{turb} = \sum_{\mathbf{k}\omega} |\tilde{v}_{E\mathbf{k}}|^2 \frac{\Delta\omega_{\mathbf{k}}}{\Delta\omega_{\mathbf{k}}^2 + \omega_{\mathbf{k}}^2}. \tag{42}$$

When $\Delta\omega_{\mathbf{k}} \gg \omega_{\mathbf{k}}$, this diffusion coefficient corresponds to a random walk with step $\tilde{v}_{Er}\tau_c$ and correlation time $\tau_{c\mathbf{k}} \sim 1/\Delta\omega_{\mathbf{k}}$.

III. ENTROPY PRODUCTION

A. Production rate of the total entropy

We use here the conventional definition of the entropy,[34] which is defined as

$$S_s = \frac{1}{\varsigma-1} \int d^3\mathbf{x}\, n_s \log(T_s n_s^{1-\varsigma}). \tag{43}$$

It is reminded that ς is the adiabatic index and must be larger than 1 ($\varsigma=5/3$ in most cases). Using Eqs. (5) and (6), the entropy time evolution is found to be given by the relation

$$\begin{aligned} \partial_t S_s = &-\frac{1}{\varsigma-1}\int d^3\mathbf{x} \frac{\mathbf{q}_\mathbf{s}^* \cdot \nabla T_s}{T_s^2} \\ &- \frac{1}{\varsigma-1} \oint \log(T_s n_s^{1-\varsigma}) \boldsymbol{\Gamma}_s \cdot \mathbf{n} d^2 S \\ &- \frac{1}{\varsigma-1} \oint \frac{1}{T_s} \mathbf{q}_\mathbf{s}^* \cdot \mathbf{n} d^2 S, \end{aligned} \tag{44}$$

where $\mathbf{n}$ is the unit vector normal to the surface that enclose the volume of integration and $\boldsymbol{\Gamma}_\mathbf{s} = n_s v_E + \boldsymbol{\Gamma}_\mathbf{s}^*$ is the total particle flux. Assuming for simplicity that the surface terms vanish (due to appropriate boundary conditions), one is left with a volume contribution proportional to the heat flux. Given the definition of the diamagnetic heat flux equation (2), it is found that the entropy production is zero, as expected since equations are nondissipative. Heat conduction would lead to an extra contribution in the heat flux of the form $\mathbf{q}_{\mathbf{s},\mathbf{cond}} = -n_s \chi_s \nabla T_s$. Equation (44) indicates that such a term produces an increase of the entropy. Going back to the collisionless case, i.e., to the case with a vanishing entropy production rate, it is expected that transport, which controls the evolution of the $m=0$, $n=0$ components of the fields, should lead to an entropy increase. This positive variation must be balanced by a decrease of the entropy associated to fluctuations. In turbulence simulations, small scales are in fact dissipated by small diffusion terms or by mesh finite size effects. These various processes lead to an overall increase of the entropy. Hence the interesting quantity is the entropy production rate associated to the "equilibrium" profiles ($m=0$, $n=0$ components of density and temperature, labeled *eq*), whose time evolution is controlled by turbulent transport. The following section is devoted to the calculation of this quantity within the frame of the quasilinear theory.

B. Production rate of the mean field entropy

This section describes the calculation of the entropy associated to the evolution of the $m=0$, $n=0$ components of the fields, i.e., of the mean fields. The mean field entropy is defined as

$$S_{s,eq} = \frac{1}{\varsigma-1} \int d^3\mathbf{x}\, n_{s,eq} \log(T_{s,eq} n_{s,eq}^{1-\varsigma}). \tag{45}$$

The element of volume integral $dV = d^3\mathbf{x}$ in Eq. (45) is given by the relation

$$dV = I_s dV_{cyl}, \tag{46}$$

where $dV_{cyl} = 4\pi^2 R_0 r dr$ is the volume element in a cylinder. The demonstration for trapped (resp., passing) particles is given in Appendix A (resp., Appendix B). Details can also be found in Refs. 11 and 35. The difference between the actual volume integration and the cylindrical expression reflects the geometrical nature of the curvature pinch, which acts through compressibility terms in the density equation. Regarding this question, it must be noted that the continuity equation Eq. (5) would not satisfy the constraint of particle conservation if the wrong element of volume integration would be used.

Assuming zero fluxes at the plasma edge, one finds

$$\partial_t S_{s,eq} = -\int dV n_{s,eq}\left\{ \frac{\Gamma_s}{\bar{n}_{s,eq}} \frac{\partial_r \bar{n}_{s,eq}}{\bar{n}_{s,eq}} + \frac{Q_s}{\bar{n}_{s,eq}\bar{T}_{s,eq}} \frac{\partial_r \bar{T}_{s,eq}}{\bar{T}_{s,eq}} \right\}. \tag{47}$$

The latter form gives the fluxes and thermodynamical forces[36] in the following way:

$$\frac{\Gamma_s}{\bar{n}_{s,eq}} = -\frac{\delta(\partial_t S_{s,eq})}{\delta\left(\dfrac{\partial_r \bar{n}_{s,eq}}{\bar{n}_{s,eq}}\right)}, \quad \frac{Q_s}{\bar{n}_{s,eq}\bar{T}_{s,eq}} = -\frac{\delta(\partial_t S_{s,eq})}{\delta\left(\dfrac{\partial_r \bar{T}_{s,eq}}{\bar{T}_{s,eq}}\right)}, \tag{48}$$

where the symbol δ indicates a functional derivative. Calculating the entropy production with nonnormalized profiles or other quantities (e.g., pressure instead of temperature) would have led to a less compact and symmetrical form. It is stressed here that the thermodynamical forces are *not* the logarithmic gradient of the density and temperature, but those of the normalized profiles $\bar{n}_{s,eq}$ and $\bar{T}_{s,eq}$.

Using a diagonal representation of the fluxes Eq. (39), one finds the following expression:

$$\partial_t S_{s,eq} = \int dV n_{s,eq} D_{turb}\left\{ \left(\frac{\partial_r \bar{n}_{s,eq}}{\bar{n}_{s,eq}}\right)^2 + \frac{1}{\varsigma - 1}\left(\frac{\partial_r \bar{T}_{s,eq}}{\bar{T}_{s,eq}}\right)^2 \right\}. \tag{49}$$

The expression (49) is the (positive) quantity expected for a pure diffusive system. In the general case, the quasilinear fluxes Eq. (37) lead to the following expression of the entropy production rate, written in a matrix form:

$$\partial_t S_{s,eq} = \int dV n_{s,eq} \sum_{\mathbf{k}\omega} i \frac{|\tilde{v}_{E,\mathbf{k}\omega}|^2}{D_{s,\mathbf{k}\omega}} \left(\frac{\partial_r \bar{n}_{s,eq}}{\bar{n}_{s,eq}}, \frac{\partial_r \bar{T}_{s,eq}}{\bar{T}_{s,eq}} \right) \cdot \begin{bmatrix} \omega - (2\varsigma - 1)\omega_{ds} & \omega_{ds} \\ \omega_{ds} & \dfrac{\omega - \omega_{ds}}{\varsigma - 1} \end{bmatrix} \cdot \begin{pmatrix} \dfrac{\partial_r \bar{n}_{s,eq}}{\bar{n}_{s,eq}} \\ \dfrac{\partial_r \bar{T}_{s,eq}}{\bar{T}_{s,eq}} \end{pmatrix}. \tag{50}$$

An alternative expression of Eq. (50), reminiscent of the diffusive rate of production (49) is

$$\partial_t S_{s,eq} = \int dV n_s \sum_{\mathbf{k}\omega} i |\tilde{v}_{E,\mathbf{k}\omega}|^2 \left\{ \frac{1}{\omega - \omega_{ds}} \left(\frac{\partial_r \bar{n}_{s,eq}}{\bar{n}_{s,eq}}\right)^2 + \frac{1}{\varsigma - 1}\frac{\omega - \omega_{ds}}{D_{s,\mathbf{k}\omega}} \left(\frac{\partial_r \bar{T}_{s,eq}}{\bar{T}_{s,eq}} + \frac{(\varsigma - 1)\omega_{ds}}{\omega - \omega_{ds}} \frac{\partial_r \bar{n}_{s,eq}}{\bar{n}_{s,eq}} \right)^2 \right\}. \tag{51}$$

Positivity is not yet demonstrated at this stage. This is done by noting that the resonant function $1/D_{s,\mathbf{k}\omega}$ exhibits two poles

$$\frac{\omega_\pm}{\varsigma \omega_{ds}} = 1 \pm \delta, \quad \delta = \sqrt{1 - 1/\varsigma}, \tag{52}$$

or equivalently

$$\frac{1}{D_{s,\mathbf{k}\omega}} = \frac{1}{\omega_+ - \omega_-}\left\{ \frac{1}{\omega - \omega_+} - \frac{1}{\omega - \omega_-} \right\}. \tag{53}$$

The structure of Eq. (50) shows that the poles which must be counted in the summation over the frequencies in the expression Eq. (51) are the zeros $\omega_\pm$ of $D_{s,\mathbf{k}\omega}$ [i.e., the pole $\omega = \omega_{Ds}$ is irrelevant in Eq. (51)]. Hence

$$\partial_t S_{s,eq} = \frac{1}{\varsigma - 1}\pi \int dV n_{s,eq} \sum_{\mathbf{k},\varepsilon=\pm 1} |\tilde{v}_{E,\mathbf{k}\omega_\varepsilon}|^2 \varepsilon \frac{\omega_\varepsilon - \omega_{ds}}{\omega_+ - \omega_-} \times \left(\frac{\partial_r \bar{T}_{s,eq}}{\bar{T}_{s,eq}} + \frac{(\varsigma - 1)\omega_{ds}}{\omega_\varepsilon - \omega_{ds}} \frac{\partial_r \bar{n}_{s,eq}}{\bar{n}_{s,eq}} \right)^2. \tag{54}$$

From the expression (45) of the frequencies $\omega_\pm$, it appears that $(\omega_\varepsilon - \omega_{ds})\varepsilon(\omega_+ - \omega_-) > 0$, provided that the adiabatic index satisfies the condition $\varsigma > 1$. Thus the entropy production rate is always positive, as expected. This result applies whatever the details of the fluctuation spectrum. It relies nevertheless on the assumptions for using the quasilinear theory.[37]

We now consider the case where the fluctuation frequency spectrum is given by Eq. (41). The entropy production rate is then given by

$$\partial_t S_{s,eq} = \int dV n_{s,eq} \left(\frac{\partial_r \bar{n}_{s,eq}}{\bar{n}_{s,eq}}, \frac{\partial_r \bar{T}_{s,eq}}{\bar{T}_{s,eq}} \right) \cdot \begin{bmatrix} D_{s,nn} & D_{s,nT} \\ D_{s,nT} & D_{s,TT} \end{bmatrix} \cdot \begin{pmatrix} \dfrac{\partial_r \bar{n}_{s,eq}}{\bar{n}_{s,eq}} \\ \dfrac{\partial_r \bar{T}_{s,eq}}{\bar{T}_{s,eq}} \end{pmatrix}, \tag{55}$$

where

$$D_{s,nn} = \pi \sum_{\mathbf{k}\varepsilon=\pm 1} |\tilde{v}_{E\mathbf{k}\omega_\varepsilon}|^2 \frac{1 - \varepsilon\delta}{2},$$

$$D_{s,TT} = \pi \frac{1}{\varsigma - 1} \sum_{\mathbf{k}\varepsilon=\pm 1} |\tilde{v}_{E\mathbf{k}\omega_\varepsilon}|^2 \frac{1 + \varepsilon\delta}{2},$$

$$D_{s,nT} = \pi \frac{1}{\varsigma - 1} \sum_{\mathbf{k}\varepsilon=\pm 1} |\tilde{v}_{E\mathbf{k}\omega_\varepsilon}|^2 \frac{\varepsilon\delta}{2}. \tag{56}$$

The entropy production rate Eq. (55) is positive, as shown by the general expression Eq. (54). This can also be verified by looking at the structure of Eq. (55). The matricial product is positive if one of the two diagonal terms is positive, i.e., $D_{s,nn} > 0$ or $D_{s,TT} > 0$, and if the determinant is positive, i.e., $D_{s,nn} D_{s,TT} - D_{s,nT}^2 > 0$. $D_{s,nn}$ and $D_{s,TT}$ are obviously positive since $0 < \delta < 1$. Moreover, the determinant is given by

$$D_{s,nn}D_{s,TT}-D_{s,nT}^2=\frac{1}{\varsigma-1}\prod_{\varepsilon=\pm1}\pi\sum_{\mathbf{k}}|\tilde{v}_{E\mathbf{k}\omega_\varepsilon}|^2. \quad (57)$$

Thus it is also positive. Equation (54) shows that the entropy production rate is a positive quadratic form of the logarithmic gradients of $\bar{n}_{s,eq}$ and $\bar{T}_{s,eq}$. Hence it is minimal when the density and temperature profiles coincide with the canonical profiles given by Eqs. (14).

IV. KINETIC APPROACH

A. Production rate of the total entropy

It turns out that the entropy is a quantity much easier to manipulate in kinetic than fluid theory. Starting from the usual definition

$$S_s=-\int d^3\mathbf{x}d^3\mathbf{p}F_s\log(F_s), \quad (58)$$

where $F_s(\mathbf{x},\mathbf{p})$ is the distribution function of particles of the species s. Obviously, if F_s is solution of the Vlasov equation $\partial_t F_s-[H_s,F_s]=0$, the entropy production rate vanishes (H_s is the Hamiltonian). As noticed previously, small-scale dissipation or collisions will in fact lead to an overall increase of entropy. Here we are interested in the increase of the "mean field" entropy, defined as

$$S_{s,eq}=-\int d^3\mathbf{x}d^3\mathbf{p}F_{s,eq}\log(F_{s,eq}). \quad (59)$$

In a tokamak (or any other integrable quasiperiodic system), particles trajectories can be described in terms of a system of action-angle variables $(\boldsymbol{\alpha},\mathbf{J})$. The details of the set of variables in a tokamak is described elsewhere[38,39] and will not be repeated here. In this frame, the quasilinear theory can be written in a compact form, i.e.,

$$\partial_t F_{s,eq}=\partial_{\mathbf{J}}\mathbf{D_s}\partial_{\mathbf{J}}F_{s,eq}, \quad (60)$$

where

$$D_{s,kl}=\pi\sum_{\mathbf{n}\omega}n_kn_l|h_{\mathbf{n}\omega}|^2\delta(\omega-\mathbf{n}\cdot\boldsymbol{\Omega}_s), \quad (61)$$

where $\Omega_{s,k}=\partial_{J_k}H_{s,eq}$ are the three angular frequencies of the unperturbed Hamiltonian, and the $h_{\mathbf{n}\omega}$'s are the components of the perturbed Hamiltonian

$$\delta H=\sum_{\mathbf{n}\omega}h_{\mathbf{n}\omega}e^{i(\mathbf{n}\cdot\boldsymbol{\alpha}-\omega t)}. \quad (62)$$

The production rate of the entropy is then found to be

$$\partial_t S_{s,eq}=\pi\int d^3\mathbf{J}(2\pi)^3F_{s,eq}\sum_{\mathbf{n}\omega}|h_{\mathbf{n}\omega}|^2\delta(\omega-\mathbf{n}\cdot\boldsymbol{\Omega}_s) \times[\mathbf{n}\cdot\partial_{\mathbf{J}}\log(F_{s,eq})]^2. \quad (63)$$

It is always positive. Hence the demonstration of a positive increase of the mean field entropy is straightforward. The detailed calculation for passing and trapped particles has been given elsewhere.[35] Simplified expressions are given in Appendix B. For passing particles, flute modes and an equilibrium distribution function of the form

$$F_{s,eq}=\frac{n_{s,eq}}{(2\pi m_s)^{3/2}T_{s\perp,eq}T_{s\|,eq}^{1/2}}e^{-(m_sv_\|^2/T_{s\|,eq})-(\mu B_{eq}/T_{s\perp,eq})} \quad (64)$$

the entropy production rate reads

$$\partial_t S_{s,eq}=\pi\int dVn_{s,eq}\int_{-\infty}^{+\infty}\frac{d\zeta}{\sqrt{2\pi}}e^{-(1/2)\zeta^2}\int_0^{+\infty}dhe^{-h} \times\sum_{\mathbf{k}\omega}|v_{E,\mathbf{k}\omega}|^2\delta\left(\omega-\frac{1}{2}\omega_{ds}(\zeta^2+h)\right)\left\{\frac{\partial_r\bar{n}_{s,eq}}{\bar{n}_{s,eq}} +\left(\frac{1}{2}\zeta^2-\frac{1}{2}\right)\frac{\partial_r\bar{T}_{s\|,eq}}{\bar{T}_{s\|,eq}}+(h-1)\frac{\partial_r\bar{T}_{s\perp,eq}}{\bar{T}_{s\perp,eq}}\right\}^2, \quad (65)$$

where

$$\frac{\partial_r\bar{n}_{s,eq}}{\bar{n}_{s,eq}}=\frac{\partial_r n_{s,eq}}{n_{s,eq}}+\frac{2}{R_0},\quad \frac{\partial_r\bar{T}_{s\|,eq}}{\bar{T}_{s\|,eq}}=\frac{\partial_r T_{s\|,eq}}{T_{s\|,eq}}+\frac{2}{R_0}, \quad \frac{\partial_r\bar{T}_{s\perp,eq}}{\bar{T}_{s\perp,eq}}=\frac{\partial_r T_{s\perp,eq}}{T_{s\perp,eq}}+\frac{1}{R_0}. \quad (66)$$

In the limit of large frequencies, it reduces to

$$\partial_t S_{s,eq}=\int dVn_{s,eq}D_{turb}\left\{\left(\frac{\partial_r\bar{n}_{s,eq}}{\bar{n}_{s,eq}}\right)^2+\frac{1}{2}\left(\frac{\partial_r\bar{T}_{s\|,eq}}{\bar{T}_{s\|,eq}}\right)^2 +\left(\frac{\partial_r\bar{T}_{s\perp,eq}}{\bar{T}_{s\perp,eq}}\right)^2\right\}. \quad (67)$$

This expression coincides with Eq. (43) when $\bar{T}_{s\|,eq}=\bar{T}_{s\perp,eq}=\bar{T}_{s,eq}$, for an adiabatic index $\varsigma=5/3$, and a diffusion coefficient that is given by Eq. (40). It must be noted however that the condition $\bar{T}_{s\|,eq}=\bar{T}_{s\perp,eq}$ is by no means obvious, as it leads to different parallel and perpendicular non-normalized temperature profiles. It is nevertheless consistent with a gyro-Landau fluid calculation that differentiates between parallel and perpendicular temperatures, as in the GLF23 transport model.[40]

For trapped electrons, the entropy production rate reads (Appendix B)

$$\partial_t S_{s,eq}=\pi\int dVn_{s,eq}\frac{2}{\sqrt{\pi}}\int_0^{+\infty}dEE^{1/2}e^{-E}\sum_{\mathbf{k}\omega}|v_{E,\mathbf{k}\omega}|^2 \times\delta\left(\omega-\omega_{ds}\frac{2E}{3}\right)\left\{\frac{\partial_r\bar{n}_{s,eq}}{\bar{n}_{s,eq}}+\left(E-\frac{3}{2}\right)\frac{\partial_r\bar{T}_{s,eq}}{\bar{T}_{s,eq}}\right\}^2, \quad (68)$$

where

$$\frac{\partial_r\bar{n}_{s,eq}}{\bar{n}_{s,eq}}=\frac{\partial_r n_{s,eq}}{n_{s,eq}}+\frac{2}{R_0}\lambda_s,\quad \frac{\partial_r\bar{T}_{s,eq}}{\bar{T}_{s,eq}}=\frac{\partial_r T_{s,eq}}{T_{s,eq}}+\frac{2}{R_0}(\varsigma-1)\lambda_s. \quad (69)$$

In the limit of large frequencies, it reduces to

$$\partial_t S_{s,eq} = \int dV n_{s,eq} D_{turb} \left\{ \left(\frac{\partial_r \bar{n}_{s,eq}}{\bar{n}_{s,eq}} \right)^2 + \frac{1}{\varsigma - 1} \left(\frac{\partial_r \bar{T}_{s,eq}}{\bar{T}_{s,eq}} \right)^2 \right\}. \tag{70}$$

This result is identical to Eq. (49). Hence it appears both fluid and kinetic calculations give the same result in the limit of large frequencies. It is stressed however that the precession frequency was assumed to depend weakly on the pitch-angle variable for this calculation. This assumption is valid for values of the magnetic shear of the order of unity, but may lead to significant differences for low or negative shear.[41] In the general case, the kinetic expression Eqs. (65) and (68) must be compared to the fluid expression Eq. (50). This opens the way for developing closure schemes. This question is left for future work.

In conclusion the entropy production rate is minimum in kinetic theory when profiles coincide with the canonical profiles, as in fluid theory. This result still applies if FLR effects are included since this is equivalent to replacing the components of the perturbed Hamiltonian in Eq. (63) by their gyroaverage values. Parallel dynamics (still using an effective curvature) can also be included (see Appendix B). The entropy production rate for passing particles is then found to be

$$\begin{aligned} \partial_t S_{s,eq} = \pi \int dV n_{s,eq} \int_{-\infty}^{+\infty} \frac{d\zeta}{\sqrt{2\pi}} e^{-(1/2)\zeta^2} \int_0^{+\infty} dh e^{-h} \\ \times \sum_{\mathbf{k}\omega} |v_{E,\mathbf{k}\omega}|^2 \delta\left(\omega - \frac{1}{2}\omega_{ds}(\zeta^2 + h) - k_\parallel c_s \zeta \right) \\ \times \left\{ \frac{\partial_r \bar{n}_{s,eq}}{\bar{n}_{s,eq}} + \left(\frac{1}{2}\zeta^2 - \frac{1}{2} \right) \frac{\partial_r \bar{T}_{s\parallel,eq}}{\bar{T}_{s\parallel,eq}} \right. \\ \left. + (h-1) \frac{\partial_r \bar{T}_{s\perp,eq}}{\bar{T}_{s\perp,eq}} + \zeta \left(\frac{1}{c_s} \partial_r V_{s\parallel,eq} - \frac{k_\parallel}{k_\theta \rho_s} \right) \right\}^2, \end{aligned} \tag{71}$$

where $V_{s\parallel,eq}$ is the mean parallel velocity, $c_s = (T_{s,eq}/m_s)^{1/2}$, $\rho_s = m_s c_s / e_s B_{eq}$, and the $k_\parallel$s are parallel wave numbers. In the limit of large turbulent frequencies, low parallel wave numbers, the transport matrix is diagonal and Eq. (71) (assuming $\bar{T}_{s\parallel,eq} = \bar{T}_{s\perp,eq} = \bar{T}_{s,eq}$) becomes

$$\begin{aligned} \partial_t S_{s,eq} = \int dV n_{s,eq} D_{turb} \left\{ \left(\frac{\partial_r \bar{n}_{s,eq}}{\bar{n}_{s,eq}} \right)^2 + \frac{1}{\varsigma - 1} \left(\frac{\partial_r \bar{T}_{s,eq}}{\bar{T}_{s,eq}} \right)^2 \right. \\ \left. + \left(\frac{1}{c_s} \partial_r V_{s\parallel,eq} \right)^2 \right\}. \end{aligned} \tag{72}$$

The new term in Eq. (71) that involves the gradient of the parallel velocity is associated to turbulent viscosity effect. The shift in the parenthesis represents pinch effects due to nonzero mean parallel wave number. This occurs for instance when a perpendicular flow shear is present. This property has been mentioned several times in the literature as a possible spin-up mechanism for ions.[42,43] In this case however the mean electric potential would have to be added when calculating the entropy production rate for consistency. The same effect opens the way for current generation due to turbulence.

It is stressed that these expressions are approximate and rely on the approximation of an effective curvature. For instance they do not cover the effect of the parallel dynamics of passing electrons on particle pinch found by several authors.[6,44,45] Also it does not apply for a slab ITG turbulence.

V. PINCHES, ONSAGER SYMMETRIES, AND SECOND PRINCIPLE

A. Curvature and thermodiffusion pinches

In peculiar cases like Eq. (39), the transport matrix is found to be diagonal. However, since the diffused quantities are normalized to the canonical profiles, pinch terms appear when fluxes are expressed as functions of the usual density and pressure

$$\begin{pmatrix} \Gamma_s \\ Q_s \end{pmatrix} = - \begin{bmatrix} D_{turb} & 0 \\ 0 & \frac{1}{\varsigma - 1} D_{turb} \end{bmatrix} \cdot \begin{pmatrix} \partial_r n_{s,eq} + \frac{2}{R_0} \lambda_s n_{s,eq} \\ n_{s,eq} \partial_r T_{s,eq} + (\varsigma - 1) \frac{2}{R_0} \lambda_s n_{s,eq} T_{s,eq} \end{pmatrix}. \tag{73}$$

The corresponding pinch velocities are always pointed inward when $\lambda_s > 0$. For trapped electrons, $\lambda_s = 1/4 + 2s/3$ can be in principle negative for negative magnetic shear. However this corresponds also to a case where turbulence is either quenched or diminished. Thus this peculiarity is in practice difficult to verify experimentally. To the best of our knowledge negative shear is not observed to produce hollow density profiles.

These velocities are called "curvature" pinch velocity, as they result directly from the toroidal geometry. It is stressed once more that the curvature of the magnetic field (or equivalently the gradient of magnetic field) is not a thermodynamical force. These terms are a consequence of the geometry. From the kinetic point of view, they result from a diffusion in phase space that is constrained by invariants of motion (Lagrangian invariants). In the general case however, the transport matrix is not diagonal, leading to a thermodiffusion term in the particle flux, and its Onsager symmetrical contribution in the heat flux. This property is illustrated by the expression Eq. (37) of fluxes.

For the sake of simplicity, we assume a frequency spectrum of the form Eq. (41). Fluxes are obtained from the entropy production rate Eq. (55)

$$\begin{pmatrix} \dfrac{\Gamma_s}{\bar{n}_{s,eq}} \\ \dfrac{Q_s}{\bar{n}_{s,eq} \bar{T}_{s,eq}} \end{pmatrix} = - \begin{bmatrix} D_{s,nn} & D_{s,nT} \\ D_{s,nT} & D_{s,TT} \end{bmatrix} \cdot \begin{pmatrix} \dfrac{\partial_r \bar{n}_{s,eq}}{\bar{n}_{s,eq}} \\ \dfrac{\partial_r \bar{T}_{s,eq}}{\bar{T}_{s,eq}} \end{pmatrix}. \tag{74}$$

The diffusion coefficients $D_{s,nn}$, $D_{s,TT}$, and $D_{s,nT}$ are given in Eq. (56). We consider now the special case where the fluctuation spectrum is given by the relation Eq. (41). The exact expressions can be calculated exactly, but are rather cumbersome. They get simpler in the limit of large frequency broadening $\Delta\omega_k \gg \omega_k$, ω_{ds}, i.e.,

$$D_{s,nn} \approx D_{turb} = \sum_{\mathbf{k}} \frac{|\tilde{v}_{E\mathbf{k}}|^2}{\Delta\omega_{\mathbf{k}}},$$

$$D_{s,TT} \approx \frac{1}{\varsigma - 1} D_{turb},$$

$$D_{s,nT} \approx \pi \sum_{\mathbf{k}} \omega_{ds} \partial_\omega (|\tilde{v}_{E\mathbf{k}\omega}|^2)|_{\omega=\varsigma\omega_{Ds}}$$

$$\approx \sum_{\mathbf{k}} \frac{|\tilde{v}_{E\mathbf{k}}|^2}{\Delta\omega_{\mathbf{k}}} \frac{2(\omega_{\mathbf{k}} - \varsigma\omega_{ds})\omega_{ds}}{\Delta\omega_{\mathbf{k}}^2}, \tag{75}$$

where ς is the adiabatic index. The first line of Eq. (74) (particle flux) is already known.[22] Thus expression (74) generalizes the expression usually found in the literature to describe the particle flux

$$\Gamma_{s,n} = - n_{s,eq} D_{turb} \left[\frac{\partial_r n_{s,eq}}{n_{s,eq}} + C_{curv} \frac{2}{R_0} + C_T \frac{\partial_r T_{s,eq}}{T_{s,eq}} \right]. \tag{76}$$

The thermodiffusion coefficient C_T changes its sign depending on the sign of the average phase velocity $\langle \omega_{\mathbf{k}} / \omega_{ds} - \varsigma \rangle$. This result is independent of the approximation $\Delta\omega_k \gg \omega_k$, ω_{ds}, i.e., is true whatever the hierarchy of frequencies. The value of $\langle \omega_{\mathbf{k}} / \omega_{ds} \rangle$ is constrained by the ambipolarity constraint. Therefore thermodiffusion is expected to be directed inward for an ITG dominated turbulence and outward for a TEM turbulence.

For impurities, the curvature pinch is directed inward and does not depend on the charge number. The coefficient of thermodiffusion decreases with the charge number Z, at least as $1/Z$. It is also found that the thermodiffusion pinch velocity is outward when turbulence is of the ITG type (i.e., $\omega_k \sim \omega_{di}$ and $\omega_{di}\omega_{dZ} > 0$) and inward for TEM turbulence.

B. Onsager symmetries and second principle

Expression (76) indicates that thermodiffusion and its counterpart has the same value and sign, i.e., when the particle pinch is directed inward, there exists a heat pinch inward as well. It is quite important to note that this symmetry only appears when fluxes and forces are correctly chosen, i.e., when they satisfy Eqs. (48). Hence a situation with a strong thermodiffusion should also be characterized by a strong heat pinch due to the density gradient. The ratio of thermodiffusion to particle diffusion is of order $(\omega_{\mathbf{k}} / \Delta\omega_{\mathbf{k}})^2 \ll 1$. Moreover, a combination of ion and electron driven modes potentially lead to a small average phase velocity. These features may provide an explanation from the smallness of thermodiffusion that is found in some experiments.[23,26]

The positivity of the entropy production rate implies that the calculated pinches are consistent with the second principle. However the meaning of the canonical profiles deserves some further clarification. Equations (74) imply that zero fluxes correspond to nonzero gradients of the actual density and temperature profiles. It is stressed here that a situation where both density and temperature are equal to the canonical profiles at zero fluxes cannot be attained. The reason is that the canonical profiles, given by Eq. (14), are linearly stable, as shown by Eq. (27). Since all turbulent diffusion coefficients vanish below the stability threshold, the system cannot relax both in density and temperature to a state where $n_s = n_{s,can}$ and $T_s = T_{s,can}$. This point was already mentioned for hybrid ("ubiquitous") modes driven by ion and electrons density gradient.[46] This property can also be seen from Eq. (21), which indicates that the level of fluctuations vanish when profiles match their canonical values. In fact the conditions of applicability of the quasilinear theory require a well-developed turbulence, i.e., gradients above the stability threshold. Hence one of the profiles at least must be above its canonical value for this calculation to be valid. This implies that one flux at least must be finite. In the extreme case where all species (i.e., including passing electrons) would be described by the fluid equations (5) and (6), one would find that the canonical profiles correspond exactly to the stability threshold.

Nevertheless it is safe to say that when decreasing all fluxes, the system tries to relax towards a state where profiles match their canonical values. This property can be considered as a generalized form of profile stiffness. It suggests that transport models would take benefit from writing the fluxes as function of profiles normalized to their canonical expressions. Also the distance between the gradients and the corresponding canonical values is indicative of the distance to profile relaxation. This may provide a useful mean to analyze the experimental data.

VI. CONCLUSION

The entropy production rate has been calculated both in the fluid and kinetic regimes for an interchange driven turbulence, using the quasilinear theory. This calculation applies for trapped particle driven instabilities or passing particles driven modes in the limit of low wavelength and strong ballooning approximation. It is found that the entropy is minimum when profiles are equal to the canonical profiles, which are Lagrangian invariants of the flow. This provides a rigorous way to determine "curvature" pinches in the particle and heat fluxes. These pinches are always directed inward, except for trapped particles when the magnetic shear is negative. This study also confirms that the magnetic field curvature is not a thermodynamical force, but rather a geometrical effect, which reflects the diffusion of particles under constraints of motion invariants. The same calculation indicates that thermodiffusion tends to be a small term in the limit of strong turbulence, i.e., when the correlation time is much smaller than the curvature drift characteristic time. As for particles, the thermodynamical force associated to temperature (or pressure) is the profile of temperature normalized to its canonical value. Also the Onsager symmetry constraint is satisfied and implies that thermodiffusion must be accompanied by a symmetrical term in the heat flux, i.e., a heat pinch due to density gradient. Finally, it is found that a state where all profiles match their canonical values is not attainable since it is linearly stable, i.e., is characterized by gradients that are below or equal to the instability threshold.

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APPENDIX A: DERIVATION OF FLUID EQUATIONS FOR TRAPPED PARTICLES

The motion of trapped particles in a tokamak is characterized by three action and angle variables. The first action is proportional to the adiabatic invariant μ, i.e., $J_1=m_s\mu/e_s$, and the corresponding angle is the cyclotron phase. The second action variable is the parallel adiabatic invariant given by

$$J_2=\frac{1}{2\pi}\oint d\ell m_s v_\parallel, \tag{A1}$$

where ℓ is the abscissa along field lines. This action is associated to the bounce angle α_2. The third invariant is the kinetic momentum around the symmetry axis of the tokamak, calculated at the bounce point,

$$J_3=e_s\psi,\quad d\psi=-B_0\frac{rdr}{q(r)}. \tag{A2}$$

It is associated to the angle $\alpha_3=\varphi-q\theta$. The poloidal flux ψ is averaged along the bounce motion of the trapped particle. The Hamiltonian is

$$H_{s,eq}=\tfrac{1}{2}m_s v_\parallel^2+\mu B_{eq}. \tag{A3}$$

The Vlasov equation is given by $\partial_t F_s-[H_s,F_s]=0$, where $H_s=H_{s,eq}+e_s\phi$ is the perturbed Hamiltonian. We assume that the turbulence frequencies are much smaller than the cyclotron and bounce frequencies. This assumption is central for the TEP theory. Assuming a distribution function of the form

$$F_s=\frac{n_s}{(2\pi m_s T_s)^{3/2}}e^{-H_{s,eq}/T_s}, \tag{A4}$$

where the density and temperature are functions of $J_3=e_s\psi$, $\alpha_3=\varphi-q\theta$ and time t only (the same dependence is assumed for the perturbed potential, i.e., these are flute modes). The Vlasov equation can then be written as

$$\partial_t F_s+\omega_3\left(\partial_{\alpha_3}F_s+F_s\frac{e_s}{T_s}\partial_{\alpha_3}\phi\right)-F_s[e_s\phi,\ln(n_s)]-\left(\frac{H_{s,eq}}{T_s}-\frac{3}{2}\right)F_s[e_s\phi,\ln(T_s)]=0, \tag{A5}$$

where ω_3 is the precession frequency of trapped particles. Fluid equations are obtained by taking the moments of the Vlasov equation (A5):

$$\partial_t n_s+\left(\partial_{\alpha_3}\langle\omega_3\rangle+\langle\omega_3\rangle\frac{e_s}{T_s}\partial_{\alpha_3}\phi\right)-[e_s\phi,n_s]=0, \tag{A6}$$

$$\partial_t p_s+\left(\partial_{\alpha_3}\left\langle\frac{2H_{s,eq}}{3}\omega_3\right\rangle+\left\langle\frac{2H_{s,eq}}{3}\omega_3\right\rangle\frac{e_s}{T_s}\partial_{\alpha_3}\phi\right)-[e_s\phi,p_s]=0. \tag{A7}$$

The element of volume in the phase space is

$$F_s d\Gamma=F_s d^3\mathbf{J}d^3\boldsymbol{\alpha}=n_s(2\pi)\frac{2}{\sqrt{\pi}}\sqrt{E}e^{-E}dE\frac{d\lambda}{\bar{\omega}_2}d\alpha_3 d\psi, \tag{A8}$$

where $\lambda=\mu/H_{s,eq}$ is the pitch-angle variable (ranging between $1/B_{min}$ and $1/B_{max}$ for trapped particles), $E=H_{s,eq}/T_{s,eq}$ is the normalized energy, and the normalized bounce frequency is

$$\frac{1}{\bar{\omega}_2}=\oint\frac{d\ell}{4(1-\lambda B_{eq})^{1/2}}. \tag{A9}$$

At this stage, the magnetic configuration is still quite general. The formula above are applicable for any equilibrium magnetic field of the form $\mathbf{B_{eq}}=f(\psi)\nabla\varphi+\nabla\psi\times\nabla\varphi$.

We restrict now the calculation to a quasicylindrical equilibrium. At lowest order in inverse aspect ratio r/R_0, ω_3 is given by the relation

$$\omega_3=\frac{q}{r}\left\langle\frac{m_s v_\parallel^2+\mu B_{eq}}{e_s B_0 R_0}[\cos(\theta)+s\theta\sin(\theta)]\right\rangle_b, \tag{A10}$$

where the bracket indicates an average over the bounce motion. At lowest order in inverse aspect ratio r/R_0, one has $B_{eq}=B_0[1-(r/R_0)\cos\theta]$ and $d\ell=qR_0 d\theta$. The following results are then found:

$$\langle\omega_3\rangle=\frac{p_s}{e_s B_0 R_0}\frac{2}{\sqrt{\pi}}\int_0^\infty dE E^{3/2}e^{-E}\int_0^\pi d\theta\cos\left(\frac{\theta}{2}\right)[\cos(\theta)+s\theta\sin(\theta)]=\frac{q}{r}\frac{2}{e_s B_0 R_0}\left(\frac{1}{4}+\frac{2s}{3}\right)p_s, \tag{A11}$$

$$\left\langle\frac{2H_{s,eq}}{3}\omega_3\right\rangle=\frac{q}{r}\frac{2}{e_s B_0 R_0}\left(\frac{1}{4}+\frac{2s}{3}\right)\varsigma T_s p_s. \tag{A12}$$

Using $\partial_{\alpha_3}=-(1/q)\partial_\theta$ and $\partial_{J_3}=-(q/e_sB_0 r)\partial_r$, it can be verified that

$$-[e_s\phi,n_s]=v_E\cdot\nabla n_s,\quad -[e_s\phi,p_s]=v_E\cdot\nabla p_s, \tag{A13}$$

$$-\partial_{\alpha_3}\langle\omega_3\rangle_b+\langle\omega_3\rangle_b\frac{e_s}{T_s}\partial_{\alpha_3}\phi=-\boldsymbol{\kappa}_\mathbf{s}\cdot\left(n_s\nabla\phi+\frac{\nabla p_s}{e_s}\right), \tag{A14}$$

$$-\partial_{\alpha_3}\left\langle\frac{2H_{s,eq}}{3}\omega_3\right\rangle+\left\langle\frac{2H_{s,eq}}{3}\omega_3\right\rangle=-\varsigma\boldsymbol{\kappa}_\mathbf{s}\cdot\left(p_s\nabla\phi+\frac{1}{e_s}\nabla\frac{p_s^2}{n_s}\right), \tag{A15}$$

where $\varsigma=5/3$ is the adiabatic index. Hence Eqs. (A6) and (A7) are identical to Eqs. (5) and (6).

One has to be careful when using the volume integration. The density that is involved here is the integral of the distribution function over the two action variables, i.e.,

$$\int dJ_1 dJ_2 (2\pi)^2 F_s = n_s(J_3,\alpha_3) I_s(J_3). \tag{A16}$$

Hence, one has the relation $d^3\mathbf{x}=I_s dV_{cyl}$ where $dV_{cyl}=4\pi^2 R_0 r dr$. The expression of the Jacobian I_s can be obtained by noting that

$$\frac{dI_s}{I_s dJ_3} = -\frac{\langle\omega_3\rangle}{p_s}, \tag{A17}$$

or equivalently

$$\frac{dI_s}{I_s dr} = \frac{2}{R_0}\left(\frac{1}{4}+\frac{2s}{3}\right), \tag{A18}$$

which is consistent with Eqs. (13) and (20). An alternative formulation of Eqs. (A7) and (A8) is therefore

$$I\partial_t n_s - [e_s\phi, I n_s] = -[I,p_s], \tag{A19}$$

$$I\partial_t p_s - [e_s\phi, I p_s] = -\varsigma[I, T_s p_s]. \tag{A20}$$

These equations are equivalent to Eqs. (18) and (19) in quasicylindrical geometry.

APPENDIX B: KINETIC QUASILINEAR THEORY AND ENTROPY PRODUCTION FOR PASSING AND TRAPPED PARTICLES

Passing particles in a tokamak are characterized by three action and angle variables. The first action is the same, i.e., $J_1=m_s\mu/e_s$, and the corresponding angle is the cyclotron phase. The second action is

$$J_2 = e_s\phi_T(J_3/e_s) + \frac{1}{2\pi}\oint d\ell\, m_s v_\parallel \tag{B1}$$

where ϕ_T is the toroidal flux. At first order in r/R_0, J_2 is close to the toroidal flux times the charge, i.e., $dJ_2=e_sB_0 r dr$, associated to the poloidal angle θ. The third invariant is the kinetic momentum along the axis, associated to the coordinate φ

$$J_3 = e_s\psi + m_s R\frac{B_\varphi}{B}v_\parallel, \quad d\psi = -B_0\frac{rdr}{q(r)}. \tag{B2}$$

For passing particles far from the loss cone, the Hamiltonian is

$$H_{s,eq} = \frac{1}{2m_sR^2}(J_3 - e\Phi_T(J_2)/e)^2 + \mu B_{eq}. \tag{B3}$$

We consider the case of flute modes strongly localized on the low field side of a tokamak, i.e., $R\approx R_0+r$, $B_{eq}\approx B_0(1-r/R_0)$, and $\mathbf{k}=(k_\theta,k_\varphi)=(m/r,n/R_0)$, where $n+m/q=0$ (flute modes). The effective curvature drift is then of the form

$$\mathbf{v_{gs}} = \tfrac{1}{2}(m_s v_\parallel^2 + \mu B_{eq})\boldsymbol{\kappa}_s \tag{B4}$$

where $\boldsymbol{\kappa}_s=-2/B_0R_0\mathbf{e}_{\boldsymbol{\theta}}$. Choosing the equilibrium distribution function as a Maxwellian with two temperatures

$$F_{s,eq} = \frac{n_s}{(2\pi m_s)^{3/2}T_{s\perp,eq}T_{s\parallel,eq}^{1/2}}e^{-(m_s v_\parallel^2/T_{s\parallel,eq})-(\mu B_{eq}/T_{s\perp,eq})}. \tag{B5}$$

It is then found that

$$\begin{aligned}\mathbf{n}\cdot\partial_{\mathbf{J}}\log(F_{s,eq}) = \frac{k_\theta}{e_sB_0}\Bigg\{&\frac{\partial_r n_{s,eq}}{n_{s,eq}} + \left(\frac{m_s v_\parallel^2}{2T_{s\parallel,eq}}-\frac{1}{2}\right)\frac{\partial_r T_{s\parallel,eq}}{T_{s\parallel,eq}} \\ &+\left(\frac{\mu B_0}{T_{s\perp,eq}}-1\right)\frac{\partial_r T_{s\perp,eq}}{T_{s\perp,eq}} + \frac{1}{R_0}\left(\frac{m_s v_\parallel^2}{T_{s\parallel,eq}} \right.\\ &\left.\left.+\frac{\mu B_0}{T_{s\perp,eq}}\right)\Bigg\}\right|.\end{aligned} \tag{B6}$$

This expression can be recast in the following way:

$$\begin{aligned}\mathbf{n}\cdot\partial_{\mathbf{J}}\log(F_{s,eq}) = \frac{k_\theta}{e_sB_0}\Bigg\{&\frac{\partial_r \bar{n}_{s,eq}}{\bar{n}_{s,eq}} + \left(\frac{m_s v_\parallel^2}{2T_{s\parallel,eq}}-\frac{1}{2}\right)\frac{\partial_r \bar{T}_{s\parallel,eq}}{\bar{T}_{s\parallel,eq}} \\ &+\left(\frac{\mu B_0}{T_{s\perp,eq}}-1\right)\frac{\partial_r \bar{T}_{s\perp,eq}}{\bar{T}_{s\perp,eq}}\Bigg\},\end{aligned} \tag{B7}$$

where

$$\begin{aligned}&\frac{\partial_r \bar{n}_{s,eq}}{\bar{n}_{s,eq}} = \frac{\partial_r n_{s,eq}}{n_{s,eq}} + \frac{2}{R_0}, \quad \frac{\partial_r \bar{T}_{s\parallel,eq}}{\bar{T}_{s\parallel,eq}} = \frac{\partial_r T_{s\parallel,eq}}{T_{s\parallel,eq}} + \frac{2}{R_0}, \\ &\frac{\partial_r \bar{T}_{s\perp,eq}}{\bar{T}_{s\perp,eq}} = \frac{\partial_r T_{s\perp,eq}}{T_{s\perp,eq}} + \frac{1}{R_0},\end{aligned} \tag{B8}$$

so that the entropy production is

$$\begin{aligned}\partial_t S_{s,eq} = \pi\int dV n_{s,eq}\int_{-\infty}^{+\infty}&\frac{d\zeta}{\sqrt{2\pi}}e^{-(1/2)\zeta^2}\int_0^{+\infty}dh e^{-h} \\ &\times\sum_{\mathbf{k}\omega}|v_{E,\mathbf{k}\omega}|^2\delta\left(\omega-\frac{1}{2}\omega_{ds}(\zeta^2+h)\right)\Bigg\{\frac{\partial_r\bar{n}_{s,eq}}{\bar{n}_{s,eq}} \\ &+\left(\frac{1}{2}\zeta^2-\frac{1}{2}\right)\frac{\partial_r\bar{T}_{s\parallel,eq}}{\bar{T}_{s\parallel,eq}} + (h-1)\frac{\partial_r\bar{T}_{s\perp,eq}}{\bar{T}_{s\perp,eq}}\Bigg\}^2.\end{aligned} \tag{B9}$$

If the parallel motion of particles is taken into account, and the assumption of flute modes is relaxed, two modifications must be done. First, the resonant condition must be modified, i.e., $\mathbf{n}\cdot\partial_{\mathbf{J}}H_{eq}=k_\theta\nu_{g\theta}+k_\parallel\nu_\parallel$. Second, the gradient of the mean parallel velocity $V_{s\parallel,eq}$ must be added for consistency. This is done using the following distribution function:

$$\begin{aligned}F_{s,eq} = &\frac{n_s}{(2\pi m_s)^{3/2}T_{s\perp,eq}T_{s\parallel,eq}^{1/2}}e^{-(m_s v_\parallel^2/T_{s\parallel,eq})-(\mu B_{eq}/T_{s\perp,eq})} \\ &\times\left[1+\frac{m_s v_\parallel V_{s\parallel,eq}}{T_{s\parallel,eq}}\right].\end{aligned} \tag{B10}$$

One then finds

$$\mathbf{n}\cdot\partial_{\mathbf{J}}\log(F_{s,eq}) = \frac{k_\theta}{e_sB_0}\left\{\frac{\partial_r n_{s,eq}}{n_{s,eq}} + \left(\frac{m_s v_\parallel^2}{2T_{s\parallel,eq}} - \frac{1}{2}\right)\frac{\partial_r T_{s\parallel,eq}}{T_{s\parallel,eq}} + \left(\frac{\mu B_0}{T_{s\perp,eq}} - 1\right)\frac{\partial_r T_{s\perp,eq}}{T_{s\perp,eq}} + \frac{1}{R_0}\left(\frac{m_s v_\parallel^2}{T_{s\parallel,eq}} + \frac{\mu B_0}{T_{s\perp,eq}}\right) + \frac{m_s v_\parallel}{T_{s\parallel,eq}}\left(\partial_r V_{s\parallel,eq} - \frac{e_sB_0}{m_s}\frac{k_\parallel}{k_\theta}\right)\right\}\Bigg|, \tag{B11}$$

and the entropy production rate becomes

$$\partial_t S_{s,eq} = \pi\int dV n_{s,eq}\int_{-\infty}^{+\infty}\frac{d\zeta}{\sqrt{2\pi}}e^{-(1/2)\zeta^2}\int_0^{+\infty} dh e^{-h} \times\sum_{\mathbf{k}\omega}|v_{E,\mathbf{k}\omega}|^2\delta\left(\omega - \frac{1}{2}\omega_{ds}(\zeta^2+h) - k_\parallel c_s\zeta\right) \times\left\{\frac{\partial_r\bar{n}_{s,eq}}{\bar{n}_{s,eq}} + \left(\frac{1}{2}\zeta^2 - \frac{1}{2}\right)\frac{\partial_r\bar{T}_{s\parallel,eq}}{\bar{T}_{s\parallel,eq}} + (h-1)\frac{\partial_r\bar{T}_{s\perp,eq}}{\bar{T}_{s\perp,eq}} + \zeta\left(\frac{1}{c_s}\partial_r V_{s\parallel,eq} - \frac{k_\parallel}{k_\theta\rho_s}\right)\right\}^2. \tag{B12}$$

As for trapped particles, the volume integral is $dV = I_s dV_{cyl}$, where

$$\frac{dI_s}{I_s dJ_3} = -\frac{\langle\omega_3\rangle}{p_s}, \tag{B13}$$

or equivalently

$$\frac{dI_s}{I_s dr} = \frac{2}{R_0}. \tag{B14}$$

The system of action-angle variables for trapped particles is given in Appendix A. The precession frequency of trapped particles is of the form

$$\omega_3 = \frac{2E}{3}\omega_{ds}(\lambda). \tag{B15}$$

It turns out that the frequency $\omega_{ds}(\lambda)$ depends weakly on the pitch-angle variable λ for values of the magnetic shear of the order of unity.[40] This approximation is less acceptable for low or negative magnetic shear. Hence substantial differences from are expected in this regime. If the precession frequency is made identical to its average, one finds (Appendix A)

$$\omega_{ds} = \frac{q}{r}\frac{2}{e_sB_0R_0}\left(\frac{1}{4} + \frac{2s}{3}\right) = \frac{q}{r}\frac{2}{e_sB_0R_0}\lambda_s. \tag{B16}$$

We consider a Maxwellian of the form

$$F_{s,eq} = \frac{n_s}{(2\pi m_s T_{s,eq})^{3/2}}e^{-H_{s,eq}/T_{s,eq}}. \tag{B17}$$

It is then found that

$$\mathbf{n}\cdot\partial_{\mathbf{J}}\log(F_{s,eq}) = \frac{k_\theta}{e_sB_0}\left\{\frac{\partial_r n_{s,eq}}{n_{s,eq}} + \left(E - \frac{3}{2}\right)\frac{\partial_r T_{s,eq}}{T_{s,eq}} + \frac{2}{R_0}\lambda_s\frac{2E}{3}\right\}. \tag{B18}$$

This expression can be recast in the following way:

$$\mathbf{n}\cdot\partial_{\mathbf{J}}\log(F_{s,eq}) = \frac{k_\theta}{e_sB_0}\left\{\frac{\partial_r\bar{n}_{s,eq}}{\bar{n}_{s,eq}} + \left(E - \frac{3}{2}\right)\frac{\partial_r\bar{T}_{s,eq}}{\bar{T}_{s,eq}}\right\}, \tag{B19}$$

where

$$\frac{\partial_r\bar{n}_{s,eq}}{\bar{n}_{s,eq}} = \frac{\partial_r n_{s,eq}}{n_{s,eq}} + \frac{2}{R_0}\lambda_s, \quad \frac{\partial_r\bar{T}_{s,eq}}{\bar{T}_{s,eq}} = \frac{\partial_r T_{s,eq}}{T_{s,eq}} + \frac{2}{R_0}(\varsigma - 1)\lambda_s, \tag{B20}$$

where $\varsigma = 5/3$. Hence the entropy production rate, normalized to the fraction of trapped electrons, is

$$\partial_t S_{s,eq} = \pi\int dV n_{s,eq}\frac{2}{\sqrt{\pi}}\int_0^{+\infty} dE E^{1/2}e^{-E}\sum_{\mathbf{k}\omega}|v_{E,\mathbf{k}\omega}|^2 \times\delta\left(\omega - \omega_{ds}\frac{2E}{3}\right)\left\{\frac{\partial_r\bar{n}_{s,eq}}{\bar{n}_{s,eq}} + \left(E - \frac{3}{2}\right)\frac{\partial_r\bar{T}_{s,eq}}{\bar{T}_{s,eq}}\right\}^2. \tag{B21}$$

[1] F. Wagner and U. Stroth, Plasma Phys. Controlled Fusion **35**, 1321 (1993).
[2] J. Stober, C. Fuchs, O. Gruber, M. Kaufmann, B. Kurzan, F. Meo, H. W. Müller, F. Ryter, and ASDEX Upgrade Team, Nucl. Fusion **41**, 1535 (2001).
[3] M. Valovic, J. Rapp, J. G. Cordey *et al.*, Plasma Phys. Controlled Fusion **44**, 1911 (2000).
[4] H. Weisen, I. Furno, S. Alberti *et al.*, Nucl. Fusion **42**, 136 (2002).
[5] M. Z. Tokar, J. Ongena, B. Unterberg, and R. R. Weynants, Phys. Rev. Lett. **84**, 895 (2000).
[6] D. R. Ernst, P. T. Bonoli, P. J. Catto, W. Dorland, C. L. Fiore, R. S. Granetz, M. Greenwald, A. E. Hubbard, M. Porkolab, M. H. Redi, J. E. Rice, K. Zhurovich, and Alcator C-Mod Group, Phys. Plasmas **11**, 2637 (2004).
[7] T. C. Luce, C. C. Petty, and J. M. C. de Haas, Phys. Rev. Lett. **68**, 52 (1992).
[8] P. Mantica, G. Gorini, G. M. D. Hogeweij, N. J. Lopes Cardozo, and A. M. R. Schilham, Phys. Rev. Lett. **85**, 4534 (2000).
[9] V. V. Yancov, JETP Lett. **60**, 171 (1994).
[10] M. B. Isichenko, A. V. Gruzinov, and P. H. Diamond, Phys. Rev. Lett. **74**, 4436 (1996); M. B. Isichenko, A. V. Gruzinov, P. H. Diamond, and P. N. Yushmanov, Phys. Plasmas **3**, 1916 (1996).
[11] D. R. Baker and M. N. Rosenbluth, Phys. Plasmas **5**, 2936 (1998).
[12] V. Naulin, J. Nycander, and J. Jull Rasmussen, Phys. Rev. Lett. **81**, 4148 (1998).
[13] B. Coppi and C. Spight, Phys. Rev. Lett. **41**, 551 (1978).
[14] W. M. Tang, G. Rewoldt, and L. Chen, Phys. Fluids **29**, 3715 (1986).
[15] P. W. Terry, Phys. Fluids B **1**, 1932 (1989).
[16] R. Waltz and R. R. Dominguez, Phys. Fluids B **1**, 1935 (1989).
[17] J. Weiland and H. Nordman, Phys. Fluids B **5**, 1669 (1993).
[18] H. Nordman, J. Weiland, and A. Jarmen, Nucl. Fusion **30**, 983 (1990).
[19] C. Estrada-Mila, J. Candy, and R. E. Waltz, Phys. Plasmas **12**, 022305 (2005).
[20] T. Dannert and F. Jenko (private communication, 2005).
[21] C. Angioni, A. G. Peeters, G. V. Pereverzev *et al.*, Phys. Plasmas **10**, 3225 (2003); C. Angioni, A. G. Peeters, G. V. Pereverzev *et al.*, Phys. Rev. Lett. **90**, 205003 (2003).
[22] X. Garbet, L. Garzotti, P. Mantica, H. Nordman, M. Valovic, H. Weisen,

and C. Angioni, Phys. Rev. Lett. **91**, 035001 (2003).
[23] G. T. Hoang, C. Bourdelle, B. Pegourie *et al.*, Phys. Rev. Lett. **90**, 155002 (2003).
[24] A. Zabolotsky, H. Weisen, and TCV Team, Plasma Phys. Controlled Fusion **45**, 735 (2003).
[25] L. Garzotti, X. Garbet, P. Mantica *et al.*, Nucl. Fusion **43**, 1829 (2003).
[26] H. Weisen, A. Zabolotsky, C. Angioni *et al.*, Nucl. Fusion **45**, L1 (2005).
[27] M. Valovic, R. Budny, L. Garzotti *et al.*, Plasma Phys. Controlled Fusion **46**, 1877 (2004).
[28] H. Weisen, A. Zabolotsky, X. Garbet *et al.*, Plasma Phys. Controlled Fusion **46**, 751 (2004).
[29] G. T. Hoang, C. Bourdelle, X. Garbet *et al.*, Phys. Rev. Lett. **93**, 135003 (2004).
[30] C. Angioni, A. G. Peeters, X. Garbet, A. Manini, F. Ryter, and ASDEX Upgrade Team, Nucl. Fusion **44**, 827 (2004).
[31] F. Nguyen and A. Samain, Plasma Phys. Controlled Fusion **39**, 1197 (1997).
[32] M. B. Isichenko and V. V. Yankov, Phys. Rep. **283**, 161 (1997).
[33] S. I. Braginskii, in *Reviews of Plasma Physics*, edited by M. A. Leontovitch (Consultants Bureau, New York, 1965), Vol. I, p. 205.
[34] K. Huang, *Statistical Mechanics* (Wiley, Singapore, 1987).
[35] F. Miskane, X. Garbet, A. Dezairi, and D. Saifaoui, Phys. Plasmas **7**, 4197 (2000).
[36] L. Landau and E. Lifshitz, *Physique Statistique* (Editions Mir, Moscow, 1984).
[37] J. C. Adam, G. Laval, and D. Pesme, Ann. Phys. (Paris) **6**, 319 (1981).
[38] A. Samain, Nucl. Fusion **10**, 325 (1970).
[39] R. B. White, *Theory of Tokamak Plasmas* (Imperial College Press, London, 2001).
[40] R. E. Waltz, G. M. Staebler, W. Dorland, G. W. Hammett, M. Kotschenreuther, and J. A. Konings, Phys. Plasmas **4**, 2482 (1997).
[41] B. B. Kadomtsev and O. P. Pogutse, *Reviews of Plasma Physics*, edited by M. A. Leontovitch (Consultant Bureau, New York, 1970), Vol. 5, p. 249.
[42] R. R. Dominguez and G. M. Staebler, Phys. Fluids B **5**, 3876 (1993).
[43] P. H. Diamond, Y. B. Lebedev, Y. M. Liang *et al.*, in *Proceedings of the 15th International Conference on Plasmas Physics and Contdted Nuclear Fusion*, Seville, 1994 (IAEA, Vienna, 1995), Vol. 3, p. 323.
[44] F. Jenko, Phys. Plasmas **7**, 514 (2000).
[45] K. Hallatschek (private communication, 2002).
[46] C. Angioni (private communication, 2005).